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AMENABLE SUBGROUPS OF SEMI-SIMPLE GROUPS AND PROXIMAL FLOWS

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ABSTRACT

We present a classification of maximal amenable subgroups of a semi-simple group G. The result is that modulo a technical connectivity condition, there are precisely 2^l conjugacy classes of such subgroups of G and we shall describe them explicitly. Here l is the split rank of the group G. These groups are the isotropy groups of the action of G on the Satake–Furstenberg compactification of the associated symmetric space and our results give necessary and sufficient conditions for a subgroup to have a fixed point in this compactification. We also study the action of G on the set of all measures on its maximal boundary. One consequence of this is a proof that the algebraic hull of an amenable subgroup of a linear group is amenable.

1. This paper is a direct outgrowth of some problems posed to the author by Harry Furstenberg concerning amenable groups, and he is really a spiritual co-author of this paper. More precisely Furstenberg raised the question of determining the amenable subgroups of a semi-simple group G (or what is the same, of a connected group G) and of determining the connection between these subgroups and the action of G on its Satake (–Furstenberg) compactification and on its Glasner space (see below for the definitions). One observes for instance that for G semi-simple, any maximal compact subgroup of G and also any minimal parabolic subgroup are maximal amenable subgroups, and these groups form two conjugacy classes of such. They are also both isotropy subgroups of points in the Satake–Furstenberg compactification. Furstenberg asked if the classical conjugacy theorems for these two kinds of subgroups could be unified perhaps in terms of classifying all maximal amenable subgroups. We shall show that this can indeed be done in a rather striking way.

Furthermore R. J. Zimmer has recently verified [15] a conjecture of Furstenberg to the effect that any amenable ergodic action (cf. [14] for definitions) of a

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semi-simple group G with finite center is induced from an action of an amenable subgroup. This says also that a cocycle of a $\mathbb{Z}$-action with values in G is cohomologous to one with values in an amenable subgroup of G. As Furstenberg pointed out, this raises the question of determining the amenable subgroups of G.

We should add that starting from another point of view, the remarkable result of Tits [13] on amenable linear groups settles nearly all questions arising in a purely algebraic context. Indeed one might view some of our results as topological analogues of this result of Tits.

We should remark that the questions posed in the present paper are really questions about semi-simple Lie groups, but it is clear from their nature and from the structure theory for connected locally compact groups, that any result obtained for semi-simple groups can be immediately translated into a result valid for any connected group and that any question of the kind considered posed for connected groups, reduces to the semi-simple case immediately. Finally a semi-simple group with finite center is up to finite groups, an algebraic group, and the presence of this algebraic structure will play a very significant role for us.

Our approach to the problems outlined above will involve the use of some notions from topological dynamics and measure theory; we explain these now and indicate their connection with the study of amenable subgroups. Suppose that G is a topological group and that X is a compact Hausdorff G-space; that is, we assume given a continuous map $(g,x)\rightarrow g\cdot x$ of $G\times X$ into X which defines a homomorphism of G into the group of homeomorphisms of X. If in addition X is a compact convex subset of a locally convex topological vector space, and if $x\rightarrow g\cdot x$ is an affine map of X into itself for each g, we say that X is an affine G-space. If X and Y are (affine) G spaces their product $X\times Y$ is in a natural way an (affine) G-space. A G-space is said to be minimal if there are no proper G-invariant closed subsets, and an affine G-space is said to be irreducible if there are no proper G invariant closed convex subsets. We note that if X is a G-space, then $M(X)$, the set of regular probability measures on X, viewed as a subset of all finite regular (signed) measures on X, with the weak-* topology, becomes an affine G-space in a natural way, and X is isomorphic, as G space, to the (invariant) set of extreme points in $M(X)$.

If X is a G-space and x and y are a pair of points of X, one distinguishes (cf. [6]) two types of behavior as follows: the points x and y are *proximal* if the closure of the G orbit $G\cdot(x,y)$ of (x,y) in $X\times X$ meets the diagonal $\Delta X\subset X\times X$; the points x and y are said to be *distal* in the contrary case. Then one says the G-space X is *proximal* if every pair of points is proximal and *distal* if every pair of distinct points is distal. It is clear that these represent in a sense extreme opposites.

One sees at once that if $M(X)$ is proximal as G space, then so is X; the converse is however false, and one says [6] that a G-space X is *strongly proximal* if $M(X)$ is proximal. It is also true that if X is strongly proximal, then so is $M(X)$. It turns out for our purposes that strongly proximal G spaces are most appropriate to our problem since a topological group G is amenable iff every strongly proximal minimal G-space is reduced to a point [6].

Moreover from [6] one may associate to any topological group a universal minimal strongly proximal G-space $B(G)$. In this context 'universal' means that any other minimal strongly proximal G-space is (uniquely) a G-equivariant image of $B(G)$. Quite clearly $B(G)$ is uniquely determined by this condition. In addition the set of probability measures $M(B(G))$ on $B(G)$ forms an irreducible affine G-space which is also universal in that every irreducible affine G-space is (uniquely) an affine G-equivariant image of this space [6]. We denote this space by $M(G)$ and call it the Glasner space of G. As we noted already, G is amenable iff $B(G)$ is a point or equivalently $M(G)$ is a point. The strategy for analyzing amenable subgroups of a given non amenable group G will be to analyze the action of G and its subgroups on $M(G)$ and $B(G)$. Indeed such an analysis turns out to be interesting in its own right.

More specifically, for connected locally compact groups G the spaces $B = B(G)$ and $M = M(G)$ can be identified explicitly and shown to coincide with well known objects that arise in different contexts, cf. [6]. Indeed in this case, B is G-homogeneous so $B = G/P$ for some subgroup P. Then P contains the radical of G together with all compact normal subgroups and the center of G so one is reduced immediately to the case when G is a semi-simple adjoint group. In that case P is known [6], [9] to be precisely a minimal parabolic subgroup, or equivalently a minimal boundary subgroup. Then M consists of measures on B, and we fix a point μ in M. The first component of our analysis of the action of G and its subgroups on M will be to determine the isotropy group H_μ of μ. Somewhat surprisingly H_μ (for any measure μ whatsoever) turns out to be an algebraic subgroup of the semi-simple group G. Moreover we show it is always amenable, a fact which then can be shown to hold for any connected group G; we also show that a cocompact normal subgroup of H_μ pointwise fixes the support of μ. Corollaries of this are that a subgroup H of G has a fixed point on M iff it is amenable, and in a rather different direction one can show using this that the algebraic hull of an amenable subgroup of an algebraic group is amenable. These results are established in section 2.

The set M contains a distinguished subset E consisting of the closure in M of the set of measures on $B = G/P$ which are invariant under some maximal compact subgroup of G and the maximal normal amenable subgroup of G. This set was introduced in [5] for semi-simple G and it was shown in [9] that it coincided as G-space with the maximal Satake compactification of the Riemannian symmetric space D associated to G [12]. (For general G, we can reduce immediately to the case when G is semi-simple by factoring out a normal amenable subgroup of G.) We call this G space $E = E(G)$, the Satake–Furstenberg space (or compactification) associated to G. The structure of E as G-space is known completely, and it consists of 2^l G-orbits where l is the split rank of G, and the isotropy groups can be written down explicitly. They are amenable as can be seen either by direct observation or by the general results of section 2, and we establish that they are maximal amenable subgroups of G. We go on to show that up to a slightly technical connectivity condition, these isotropy groups constitute *all* the maximal amenable subgroups, and again subject to the same connectivity condition any amenable is contained in one of them. Thus we find that there are precisely 2^l conjugacy classes of such maximal amenable subgroups. One of these classes is the class of P, the minimal boundary subgroup or for G semi-simple, the minimal parabolic subgroup. Another one of these 2^l classes (at least if G is semi-simple) is that of the maximal compact subgroup. These two classes are in a sense the extremes and the other 2^l-2 conjugacy classes are so to speak interpolated between them. It is interesting to see that the conjugacy theorem for maximal compact subgroups, and the seemingly unrelated conjugacy theorem for minimal parabolic subgroups now emerge as special cases of one single conjugacy theorem on maximal amenable subgroups. The fact that these maximal amenable subgroups are not connected either topologically, or even as algebraic groups (in the semi-simple case), causes some mild complications, mostly in regard to formulating the theorems. Theorem 3.4 in section 3 provides a clean statement of the conjugacy result in terms of connected groups. Finally, in addition to the classification of amenable subgroups we obtain fixed point theorems giving necessary and sufficient conditions for a subgroup H of G to have a fixed point in the Satake–Furstenberg compactification.

As a matter of notation, we shall denote by H^0 the topological component of the identity of a topological group H, and shall denote by G_0 the algebraic component of the identity of an algebraic group.

2. Let G be a connected semi-simple group and let P be a minimal boundary subgroup and $G/P = B$ the maximal boundary. We are interested in the structure of the stabilizer H_μ of a measure μ on B. Since the center of G acts trivially on B, we are quite free to replace G by a group locally isomorphic to itself. It will simplify matters considerably to choose a specific such group, namely that group locally isomorphic to G which is embedded naturally in the simply connected complex group whose Lie algebra is the complexification of that of G. When we do this, G becomes the set of real points of a semi-simple algebraic group which is connected and simply connected as algebraic group [3]. Our reason for this choice is that we wish to employ techniques from the theory of algebraic groups, and a proper covering group of G will not have the structure of an algebraic group, whereas a group covered by G may only be an open subgroup of an algebraic group. In any case the results obtained for our G can be immediately translated into results valid for any G' locally isomorphic to G.

Now as is well known the minimal boundary subgroup, P is a minimal parabolic subgroup of G. We shall systematically abuse language and identify an algebraic group with its set of points in the relevant field, namely the reals). We let D be the maximal split solvable algebraic subgroup of G. Then $D = A \cdot N$ with A a maximal $\mathbf{R}$-split torus and N a maximal unipotent subgroup. We let $D^0 = A^0N$ be the topological component of the identity of D; then A^0 is topologically a vector group and we have an Iwasawa decomposition $G = KA^0N$ where K is a maximal compact subgroup of G given as the fixed points of a Cartan involution θ such that $\theta(a) = a^{-1}$ for $a \in A$.

We begin with a result on the action of one parameter groups in D; for this, recall that a group action on a Borel space is smooth if the orbit space is smooth, i.e. is a countably separated Borel space [8].

PROPOSITION 2.1. *Let $x(t)$ be a one parameter subgroup of D. Then the action of this one parameter group on $B = G/P$ is smooth.*

PROOF. If ρ is any linear representation of G on a vector space, V, then $\rho(D)$ is a split solvable algebraic group in $\mathrm{GL}(V)$. Hence all the eigenvalues of $\rho(x(t))$ on V are real. Now we let $\mathbf{R}$ act on V by $\varphi(s)v = \exp(s)v$. Then obviously all eigenvalues of this action are real and $(t,s) \to \rho(x(t))\varphi(s)$ defines an action of $\mathbf{R} \times \mathbf{R}$ with all real eigenvalues. By a result of Pukanszky [11] such an action of a type E solvable group on a vector space is smooth. Since the orbit space of the action of $\mathbf{R} \times \mathbf{R}$ is the same as the orbit space of the induced action of $\rho(x(\cdot))$ on the space of orbits of $\varphi(\cdot)$, the latter orbit is also smooth. But this orbit space (minus the origin) is simply the orbit space of $\rho(x(t))$ acting on the projective space $P(V)$ of V, and we conclude that this orbit space is smooth. However $B = G/P$ can be G-equivariantly embedded in $P(V)$ for a suitable choice of ρ and V, (cf. [7], p. 80), and this establishes the result since an invariant subset of a smooth action is smooth.

We shall also need the same result for a single element rather than a one parameter subgroup. For this, we first observe that any $g \in D^0$ may be embedded on a one parameter group (in fact a unique one) since D^0 is a type E solvable group. We need the following simple observation.

Proposition 2.2. *If $g \in D^0$, the algebraic hull of $\{g^n\}$ is connected as algebraic group.*

Proof. Let H be the algebraic hull in question and let H^0 be its topological component of the identity. Then $g^n \in H^0$ for some n and so g^n lies on a one parameter group contained in H^0 since H^0 is abelian. But this one parameter subgroup must be the same as the one containing g and hence $g \in H^0$, and consequently H is connected as algebraic group.

Proposition 2.3. *If $g \in D^0$ and $b \in B$, and b is a periodic point under g, then b is fixed by g.*

Proof. The stabilizer of b in G is algebraic, being a conjugate of P, and hence if g^n fixes b for some n, the algebraic hull H_n of the group generated by g^n is contained in P. But the algebraic hull H_1 of the group generated by g contains H_n as a subgroup of finite index (dividing n), but as H_1 is connected as algebraic group $H_1 = H_n \subset P$ and gives b as desired.

Proposition 2.4. *If $x(t)$ is a one parameter subgroup in D, and if B_1 is the set of fixed points of this one parameter group in B, then the action of $x(t)$ on $B - B_1$ is free.*

Proof. If not, there would be a periodic orbit with $x(a/2)\cdot b \neq b$ but $x(a)\cdot b = b$, contrary to 2.3.

Proposition 2.5. *If $g \in D^0$, then the integer action on B defined by powers of g is smooth; moreover off the fixed point set, the action is free.*

PROOF. We find a one parameter group $x(t)$ in D with $x(1) = g$. Then on the fixed point set B_1 of $x(t)$, the integer action is trivially smooth. On the complement, the action is free and smooth. By general theorems (cf. [1], p. 76) we can find a Borel cross section S for the orbits in $B - B_1$ so that $(t, s) \to x(t) \cdot s$ is a Borel isomorphism of $S \times \mathbf{R}$ onto $B - B_1$. We then set $S_1 = S \times [0, 1)$ and observe that this is a Borel cross section for the integer action on $B - B_1$ given by $g = x(1)$. Hence the orbit space of this action is smooth. The last statement follows at once from 2.4.

We are now ready to apply these considerations to the study of measures on B invariant under certain subgroups of G.

PROPOSITION 2.6. *If μ is a finite measure on B invariant under an element g which belongs to a conjugate of D^0, then the support of μ is pointwise fixed by the algebraic hull of $\{g^n\}$.*

PROOF. We simply decompose the measure μ into an integral of its ergodic components μ^y under the action of $\{g^n\}$. These ergodic components are almost all finite and invariant measures. But now since the action of $\{g^n\}$ is smooth, these μ^y are transitive, that is they are each concentrated on a single orbit of the action, [8]. But by 2.5, all orbits outside the fixed point set are infinite and hence cannot carry a finite invariant measure. Hence almost all μ^y are point masses concentrated on a fixed point of the action. Thus μ is concentrated and hence also supported on the fixed point set B_1 since the latter is closed. However the set of all h in G fixing B_1 pointwise is an algebraic subgroup (being an intersection of conjugates of P). Hence if g fixes B_1, so does the algebraic hull of the group generated by g, and we are done.

Now let μ be any finite measure on B, and let H_μ be its isotropy group in G. Our main result in this section is the following.

THEOREM 2.7. *The group H_μ is an algebraic subgroup of G and is amenable. Moreover the maximal split solvable subgroup D_μ of H_μ pointwise fixes the support of μ. (Note that H_μ / D_μ is compact.)*

PROOF. If $h \in H_\mu$, let $L(h)$ be the algebraic hull of the group generated by h. Then $L(h)$ is abelian, and its set of real points can be decomposed as a product $L(h) = K(h)D(h)^0$ where $K(h)$ is the maximal compact subgroup of $L(h)$ and where $D(h)^0$ is the topological component of the identity of the maximal split subgroup $D(h)$ of $L(h)$. Now let $h = h_k h_d$ be the decomposition of h in this factorization. Then h_d^0, up to conjugacy, lies in some D^0, and its algebraic hull is $D(h)$.

We first average the finite measure μ over the compact subgroup $K(h)$ to obtain a measure ν which is now invariant under $K(h)$ and of course is also invariant under h as h commutes with $K(h)$. But then ν is invariant under h_d also, and Proposition 2.6 applies, and says that the algebraic hull of h_d, namely $D(h)$ must pointwise fix the support of ν. But now the averaging process of going from μ to ν can only increase the support of a measure (they are positive measures) and hence $\mathrm{supp}(\mu) \subset \mathrm{supp}(\nu)$ and the former is pointwise fixed by $D(h)$. In particular $D(h)$ fixes the measure μ so $D(h) \subset H_\mu$.

But now $h_k = hh_d^{-1}$ also fixes μ and we claim that if M is the topological closure of the group generated by h_d, then $M \cdot D(h) = L(h)$. To see this, we note that M, being compact, is algebraic and then $M \cdot D(h)$ is the set of real points of an algebraic subgroup containing h and hence must contain $L(h)$. Finally since H_μ is closed, and contains h_k, it must contain M and hence all of $L(h)$.

The following fact is now clearly relevant.

PROPOSITION 2.8. *Let G be (the real points of) a linear algebraic group defined over the reals whose real points are Zariski dense, and let H be a closed subgroup (in the Euclidean topology) of G such that for each $h \in H$, the algebraic hull of the group generated by h is contained in H. Then H is (the set of real points of) an algebraic subgroup of G.*

In order to preserve the continuity of the argument we shall defer the proof for a while. The proposition applied to $H_\mu \subset G$ yields the first assertion of the theorem that H_μ is (the set of real points of) an algebraic subgroup of G. Since any split solvable subgroup of H_μ and in particular a maximal one D_μ can be embedded in maximal split solvable subgroup of G, Proposition 2.6 says that D_μ pointwise fixes the support of μ. It is easy to see now that H_μ is amenable for if not, a Levi factor of H_μ would be noncompact and by 9.4 of [3] and the Jacobson–Morosov theorem, H_μ would contain an algebraic group M locally isomorphic to SL_2. The maximal split solvable subgroups of M must by the above fix pointwise the support of μ, and since M is generated by these subgroups, M fixes pointwise the support of μ. In particular M has a fixed point on $B = G/P$ and hence M is contained in a minimal parabolic subgroup of G, which is clearly impossible. Now all the claims of the theorem are proved, although we should remark that once H_μ is amenable, the maximal split solvable subgroup D_μ is unique and normal, and of course cocompact in H_μ.

Let us now give the proof of Proposition 2.8. We may clearly assume that G is the algebraic hull of the given subgroup H. As in the proof of the theorem, let $L(h)$ be the algebraic hull of an $h \in H$ and let $L(h)_0$ be the algebraic connected component of the identity of $L(h)$. Finally let M be the smallest algebraic subgroup of G containing all the $L(h)_0$, and let N be the actual subgroup of the set of real points of M generated by the real points of the groups $L(h)_0$ (for once it is necessary to distinguish between an algebraic group and its set of real points).

Now it is standard (see proposition 7.5 of [7]) that one can find a finite set of h's so that the map $L(h_1)_0 \times \cdots \times L(h_n)_0 \to M$ is surjective as an algebraic map. It follows immediately by the open mapping theorem that the subgroup generated by the real points of $L(h_i)_0$ contains a neighborhood of the identity and hence in particular that N is open in M. But M is a (connected) algebraic subgroup and has only a finite number of topological components and hence N is of finite index in M.

Our hypothesis that $L(h) \subset H$ implies at least now that $N \subset H$. We claim now that M is a normal algebraic subgroup of G since the elements of H permute the groups $L(h)_0$ under conjugation and consequently H must normalize the algebraic group they generate, namely M. But now H is Zariski dense in G and if H normalizes an algebraic subgroup M so must G itself. Now the product $M \cdot H$ is a group and we claim it is topologically closed since $M \cap H$ is cocompact (in fact cofinite in M by the above) and consequently the projection H_1 of $M \cdot H$ into the algebraic quotient group G/M is a closed subgroup. Now if $h \in H$, some power of h will lie in $L(h)_0$ and hence in M. This says exactly that H_1 is a torsion group. But G/M is a linear algebraic group and H_1 is a closed subgroup. By the variant of Burnside's theorem in [9] it follows that H_1 is in fact finite. Finally since H is Zariski dense in G, H_1 is Zariski dense in G/M and this implies that G/M is finite.

Since any subgroup of G containing M is algebraic it suffices simply to show that $H \supset M$; but now $H \cap M$ has the same property relative to M that H had relative to G and hence we may assume additionally that $G = M$ is in fact a connected algebraic group. We know already that H is of finite index in M and hence contains the topological component of the identity M^0 of M. If T is a maximal $\mathbf{R}$ split torus of M, we can find an element h in T^0 (the topological component of the identity) such that $L(h) = T$. Then the set of real points of T is cccontained in H. But $M = T \cdot M^0$ (cf. 14.4 of [3]) and hence $H = M$ and we are done.

Let us close this section with a few comments and addenda to the main theorem (2.7). It was formulated for semi-simple groups which were the real points of a complex simply connected group. For the record one should note the facts for a general semi-simple group.

THEOREM 2.9. *Let G be connected semi-simple Lie group of its Lie algebra and $B = G/P$ a maximal boundary and M the Glasner space of G. If $\mu \in M$, the isotropy group H_μ of μ contains the center of G and is amenable; its projection into the adjoint group G^* of G is the intersection of an algebraic subgroup of* $\mathrm{Aut}(g)$, *with G^*. The inverse image of a maximal split solvable subgroup of its image in G^* pointwise fixes the support of μ.*

As noted already such results trivially give statements valid for any locally compact connected group such as the following.

THEOREM 2.10. *Let G be connected locally compact, $B = G/P$ the maximal boundary of G and M the Glasner space of G. Then a closed subgroup H of G has a fixed point in M iff H is amenable. Moreover, if $\mu \in M$, and H_μ is its isotropy group, a cocompact normal subgroup of H_μ, pointwise fixes the support of μ and consequently the action of H_μ on the support of μ is equicontinuous.*

This formulation demonstrates well the point that the only invariant measures μ on B for a subgroup of G are so to speak the obvious ones. It is interesting to note that while the action of G on B is strongly proximal, the action of a subgroup H on the support of any measure μ on B fixed by H is as far from being strongly proximal as it can be, namely it is equicontinuous.

Finally we note that the main theorem can be used to give a simple and quick proof of a result on amenable groups that will be needed in the next section.

THEOREM 2.11. *Let V be a finite dimensional real vector space, H be a topological group, i a continuous injection of H into* $\mathrm{GL}(V)$. *If H is amenable then the algebraic hull in* $\mathrm{GL}(V)$ *of $i(H)$ is also amenable.*

PROOF. We may as usual embed $\mathrm{GL}(V)$ into $\mathrm{SL}(V')$ with $\dim V' = \dim V + 1$ and assume $i(H) \subset \mathrm{SL}(V)$. Now if M is the Glasner space of $\mathrm{SL}(V)$, amenability of H implies that $i(H)$ has a fixed point μ in M. Thus $i(H) \subset H_\mu$ the isotropy group of μ and by 2.7, H_μ is amenable and algebraic. Now the algebraic hull of $i(H)$ is a closed subgroup of H_μ, and is therefore amenable, and we are done.

This result is very close to a result of Tits [13]. Tits' result is that if H is amenable (as abstract group) and $H \subset \mathrm{GL}(V)$, V a vector space over a field of characteristic zero, then H is a finite extension of a solvable group. It is immediate that its algebraic hull $\bar{H}$ in $\mathrm{GL}(V)$ is also a finite extension of a solvable group, hence amenable. Our result 2.11 does not follow absolutely immediately as our H was taken to be amenable as topological group. However Tits' result plus some extra lines does provide another proof of 2.11 as follows.

ALTERNATE PROOF OF THEOREM 2.11. By the definition of amenability in terms of fixed points for affine actions, it is immediate that the topological closure of $i(H)$ in $\mathrm{GL}(V)$ with the relative topology is also amenable so we can assume that H is a closed subgroup of $\mathrm{GL}(V)$ with the relative topology. If H^0 is the topological connected component of the identity it too is amenable, hence it is a compact extension of a solvable group. It then follows easily that the algebraic hull $\overline{(H^0)}$ of H^0 is also a compact extension of a solvable group, hence amenable. If $\bar{H}$ is the algebraic hull of H, then $\bar{H}$ normalizes H_0 (look at the Lie algebra) and hence normalizes $\overline{(H_0)}$. Now $\bar{H}/\overline{(H^0)}$ is a linear group and $H/H \subset \overline{(H^0)}$ can be naturally viewed as a (Zariski dense) subgroup of it. But $H/H \cap \overline{(H^0)}$, as abstract group, is a quotient of the amenable group H and hence is amenable. Now the Tits result does apply and $H/H \cap \overline{(H^0)}$ and hence also $H/\overline{(H^0)}$ are finite extensions of solvable groups. Then $\bar{H}$ is amenable since $\overline{(H^0)}$ and $\bar{H}/\overline{(H^0)}$ are amenable.

3. As noted in the introduction the space M of probability measures on the maximal boundary $B = G/P$ of G has a distinguished subset E of considerable importance which was introduced in [5]. For the present let G be a connected semi-simple Lie group with finite center, and let K' be any maximal compact subgroup. Then K' is transitive on B and so B admits a unique K' invariant probability measure. The set of these measures as K' varies over all maximal compacts forms a single G orbit in M which can be naturally identified to the Riemannian symmetric space G/K for a fixed maximal compact subgroup of G, by identifying a point p of G/K with the unique probability measure on B invariant under the isotropy group of p. This identification is a homeomorphism and the closure of G/K viewed in this way as a subset of M is denoted by E. It was shown in [9] that this compactification of G/K is identical to the maximal Satake compactification [12] which was initially defined by quite different methods.

Our object in this section is to study the isotropy groups of points of E. In fact the structure of E as G-space is completely known and consists of 2^l G-orbits, where l is the split rank of G, and the isotropy groups associated to these orbits can computed explicitly; they of course turn out to be amenable (which we know already by Theorem 2.7). The intuitive idea is that these should be maximal amenable subgroups. This is indeed the case and moreover we propose to try to classify maximal amenable subgroups. The result is that, subject to a connectivity condition, the isotropy groups of points of E are precisely all the maximal amenable subgroups. We also determine when an amenable subgroup H of G can be embedded in one of these, or equivalently, we determine when a subgroup H of G have a fixed point in the maximal Satake compactification. The answer is in terms of the same connectivity condition.

Just as in the previous section we will take G to be the set of real points in the corresponding complex semi-simple simply connected group. Our results have easy and immediate translations to any group locally isomorphic to such a G and indeed to any connected group. As before we fix a maximal $\mathbf{R}$-split solvable group $D = AN$ with maximal $\mathbf{R}$ split torus A and a minimal parabolic $P = MAN$; we fix a maximal compact subgroup K which is the fixed point set of a Cartan involution θ with $\theta(a) = a^{-1}$ for $a \in A$. As usual a standard parabolic subgroup will be any subgroup Q containing P. Such a Q has a decomposition (cf. [3]) $Q = M_Q \cdot A_Q N_Q$ where $N_Q \subset N$ is the nilradical of Q, $A_Q \subset A$ is a maximal split torus contained in the radical of Q, and where M_Q is a θ-stable reductive group centralizing A_Q with compact center. We let $K_Q = K \cap M_Q$ which is a maximal compact subgroup of M_Q and we let $S_Q = K_Q A_Q N_Q$. From its construction this is (the set of real points of) an algebraic group which is amenable. Now while M_Q is always connected as an algebraic group, it may be disconnected as a topological group. Hence K_Q may also be disconnected as topological group and therefore also disconnected as algebraic group. Hence S_Q may also be disconnected as algebraic group. The amount of disconnectivity is somewhat limited and therein lies the key to the issue of when a subgroup of G has a fixed point on E.

Let H be (the set of real points of) a linear algebraic group over $\mathbf{R}$ whose rational points are Zariski dense and let A be a maximal $\mathbf{R}$-split torus of H, and let $Z(A)$ be its centralizer in H and let H_0 be the (algebraic) connected component of the identity.

DEFINITION. We say that H is *isotropically connected* (i-connected) if $H = H_0 \cdot Z(A)$. If H is any subgroup of GL(V), we say that H is isotropically connected if its algebraic hull $\bar{H}$ is isotropically connected.

A few comments are in order; our condition on H is that the map from $H_0 \times Z(A)$ to H is surjective as algebraic map; it is not difficult to show however using special properties of A that this is equivalent to the map being surjective on real points. The condition then is simply that each coset of the (real points of H) mod H_0 meets the centralizer of A. It is also not hard to see that this is also the same as demanding that each coset of the topological component of the identity meet $Z(A)$. It is also clear that any algebraic group H or more generally any subgroup H of $\mathrm{GL}(V)$ contains a unique maximal normal subgroup of finite index which is isotropically connected.

If H is algebraic, and as above A is a maximal **R**-split torus, let $N(A)$ be the normalizer of A so $N(A) \supset Z(A)$. It is immediate from the structure of amenable groups and properties of A that for (algebraically) connected H, amenability is equivalent to the equality $N(A) = Z(A)$. In other words H is amenable iff its "isotropic Weyl group" $N(A)/Z(A)$ is trivial. It is clear that for general algebraic H, the condition of amenability plus isotropic connectivity is the same as $N(A) = Z(A)$, i.e. again that the isotropic Weyl group is trivial. For a general subgroup $H \subset \mathrm{GL}(V)$, we say that H has trivial isotropic Weyl group if that is true for $\bar{H}$, its algebraic hull.

With these definitions we can state the main results of this section.

THEOREM 3.2. (1) *Let G be a real semi-simple group whose complex points form a simply connected group. Then the groups S_Q are maximal amenable subgroups, are isotropically connected, and self normalizing. As Q runs over 2^l ($l = \dim A$) standard parabolics, of G the S_Q are mutually non-conjugate and so are representatives of 2^l conjugacy classes of maximal amenable subgroups of G.*

(2) *An amenable subgroup H of G is contained in a conjugate of some S_Q iff H is isotropically connected. Consequently a maximal amenable subgroup of G which is isotropically connected is a conjugate of one of the S_Q's.*

A few comments are in order. If $Q = P$ is the minimal parabolic, then $S_Q = P$ is P itself. If at the other extreme $Q = G$, then $S_Q = K$ is a maximal compact subgroup. Thus the classical conjugacy theorems for minimal parabolic subgroups and for maximal compact subgroups can be viewed as special cases of the above theorem which classifies maximal amenable subgroups up to conjugacy modulo a connectivity condition. That such a connectivity condition is necessary is seen at once by considering the full normalizer $N(A)$ of the maximal **R**-split torus A. This is clearly amenable but is not isotropically connected, and is not contained in a conjugate of any S_Q.

The meaning of Theorem 3.2 is clarified and amplified when we interpret it in terms of the maximal Satake–Furstenberg compactification E (which as usual is regarded as a subset of the Glasner space of G).

THEOREM 3.3. *The set of all conjugates of the subgroups S_Q are precisely the isotropy groups for the action of G on E, and $s \to I_s$, the isotropy group of s, is a G-equivariant bijection of E into the set of conjugates of the S_Q's. Moreover a subgroup H of G has a fixed point on E iff H is amenable and isotropically connected or equivalently iff H has trivial isotropic Weyl group. In general any amenable subgroup of G fixes a finite subset of E.*

For convenience we have formulated these results for a specific choice of a group G among all groups locally isomorphic to one such. Just as in the previous section the results translate immediately into corresponding statements about amenable subgroups of any connected semi-simple group or more generally any connected locally compact group G. We shall give one such formulation, but we first need some terminology. In a connected locally compact group G we may find a normal subgroup N which is in fact amenable such that G/N is a connected semi-simple Lie group without center and without compact factors. In fact N is the unique maximal amenable normal subgroup of G. Let us agree to say that the split rank of G is the split rank of G/N and that the Satake–Furstenberg compactification E for G is Satake–Furstenberg compactification for G/N. Also G/N is finitely covered by a group G_1 satisfying the conditions of 3.2, and let us say that $H \subset G$ is isotropically connected if the inverse image in G_1 of the image of H in G/N is isotropically connected. With this language we can give the following formulation of our results.

THEOREM 3.4. *If G is connected locally compact, the maximal connected amenable subgroups fall into precisely 2^l conjugacy classes of subgroups of G where l is the split rank of G. These subgroups are the topological components of the identity of the isotropy groups of the action of G on its maximal Satake–Furstenberg compactification. Each of these subgroups is of finite index in its normalizer, and these normalizers are the isotropy groups of the action of G on E and are maximal amenable subgroups. An amenable subgroup H of G is contained in one of these normalizers, or equivalently has a fixed point on E, iff H is isotropically connected. Any amenable subgroup of G has a subgroup of finite index which is contained in one of the maximal connected amenable subgroups of G.*

All of the statements of this theorem follow immediately from Theorems 3.2 and 3.3, and we now proceed to the proofs of these theorems themselves.

First let us see that Theorem 3.3 follows at once from Theorem 3.2. One knows from [9] the orbit structure of E under the action of G; there are 2^l orbits, which are parameterized by the standard parabolics, and for each Q, S_Q is known to be the isotropy group at a specific point in the corresponding orbit. The fact that S_Q is self normalizing assures us that different points have distinct isotropy groups. The necessary and sufficient condition for H to have a fixed point in E is then just a translation of the second part of 3.2. The final statement follows since any amenable H has a subgroup H_1 of finite index which is isotropically connected and hence has a fixed point e. Then the orbit of e under H is finite and invariant.

We turn now to the proof of 3.2, and begin with the following fact.

PROPOSITION 3.5. *The group S_Q is amenable, isotropically connected, and self normalizing. It is a maximal proper subgroup of Q and any parabolic subgroup L of G which contains $(S_Q)_0$ (the algebraic component of the identity of S_Q) contains Q.*

PROOF. The group S_Q is clearly amenable and A_Q is evidently a maximal **R**-split torus. It is clear from the construction that any algebraic disconnectivity of S_Q must come from K_Q which by definition centralizes A_Q. Hence our connectivity condition is satisfied.

Now let H be the normalizer of S_Q; then H normalizes N_Q, which is the nilradical of the algebraic component of the identity of S_Q. But the normalizer of N_Q is exactly Q [2] and so $H \subset Q$. But then we can write $H = H_1 A_Q \cdot N_Q$ where $K_Q \subset H_1 \subset M_Q$. But K_Q is a maximal subgroup of M_Q so either $H = S_Q$ or $H = Q$. But Q does not normalize S_Q unless $S_Q = Q$, i.e. Q is minimal. Hence S_Q is self normalizing in all cases. We have also just shown that S_Q is a maximal proper subgroup of Q.

Now if L is any parabolic subgroup of G with $L \supset (S_Q)_0$, then by the above either $L \cap Q = Q$, which is what we want, or $(L \cap Q)_0 = (S_Q)_0$. But it is known (proposition 4.7 [3]) that the intersection of any two parabolic subgroups contains a maximal **R**-split torus A' of G; that is a group conjugate to A where the minimal parabolic P is MAN. Thus if $(L \cap Q)_0 = (S_Q)_0$, $(S_Q)_0 = K_Q^0 A_Q N_Q$ would contain some A' conjugate to A. But A_Q is a maximal **R**-split torus of $(S_Q)_0$ so A' is conjugate to a subgroup of A_Q, and consequently A is conjugate to a subgroup of A_Q. But this is absurd unless $A = A_Q$, in which case $Q = P = S_Q$ and then $L \supset Q$ trivially.

It follows from this that the S_Q are in distinct conjugary classes since if $gS_Qg^{-1} = hS_Rh^{-1}$, the unique smallest parabolic subgroup containing this group which exists by 3.4 would on the one hand be gQg^{-1} and on the other hRh^{-1}. Hence Q and R are conjugate and so $Q = R$. Moreover there are no proper containments between these subgroups or their algebraic components of the identity for if $g(S_Q)_0g^{-1} \subset h(S_R)_0h^{-1}$, it follows that $g(S_Q)_0g^{-1}$ is contained in the parabolic group hRh^{-1}. But by 3.4, the unique smallest parabolic containing $g(S_Q)_0g^{-1}$ is gQg^{-1} and so $P \subset Q \subset (g^{-1}h)R(g^{-1}h)^{-1}$. But each conjugacy class of parabolics contains only one standard one and so $(g^{-1}h)R(g^{-1}h)^{-1} = R$ and then $g^{-1}h \in R$ as R is self normalizing. Then $Q \subset R$, and $(S_Q)_0 \subset (g^{-1}h)(S_R)_0(g^{-1}h)$. But now the maximal **R**-split solvable subgroup of $(S_Q)_0$ (respectively $(S_R)_0$) is A'_QN_Q (respectively A_RN_R) which in this case is a normal subgroup of Q (respectively R). Consequently the maximal **R**-split solvable subgroup of $(g^{-1}h)(S_R)_0(g^{-1}h)^{-1}$ is precisely $A_R \cdot N_R$. But then we must have $A_Q \cdot N_Q \subset A_R \cdot N_R$. By just comparing dimensions, we see that this can only happen if $Q = R$, and then it follows easily that $(S_Q)_0 = g^{-1}h(S_R)_0(g^{-1}h)^{-1}$ and hence that $g(S_Q)_0g^{-1} = h(S_R)_0h^{-1}$. Thus there are, as asserted, no proper inclusions between the subgroups $g(S_Q)_0g^{-1}$. The same argument shows that there can be no proper inclusions between the gS_Qg^{-1}.

In order not to interrupt the flow of argument we assume for the moment that the second statement of the theorem is proved, and complete the proof of the first part of the theorem by showing that the S_Q are maximal amenable. For if $L \supset S_Q$ properly with L amenable, we may by Theorem 2.11 take L to be algebraic. If L/S_Q is finite, then as above we argue that L normalizes N_Q and hence $L \subset P$. But then $L = P$ is not amenable unless $Q = P$ and then $L = S_P = P$.

Thus we may assume that S_Q has positive codimension in L, and so L_0 the algebraic component of the identity of L properly contains $(S_Q)_0$. But L_0 is isotropically connected, and by the final statement of the theorem (which we are assuming proved) is contained in some gS_Rg^{-1} and hence in $g(S_R)_0g^{-1}$. Then $(S_Q)_0$ is properly contained in $g(S_R)_0g^{-1}$, contrary to what we have just shown, and therefore S_Q is maximal amenable.

We now turn to the second part and show that any isotropically connected amenable H is contained in some gS_Qg^{-1}. By Theorem 2.11, we may assume that H is algebraic. We may also assume that H is not contained in any amenable isotropically connected subgroup of properly larger dimension. Now let U be the unipotent radical of H, which as H is amenable is also the maximal unipotent subgroup. Then by a theorem of Borel and Tits [4] U is contained in the nilradical of some parabolic group L so that the full normalizer $N(U)$ of U is contained in L. But U is normalized by H so $H \subset L$, and after conjugating by an appropriate element of G, we may take $L = Q = M_Q \cdot A_Q \cdot N_Q$, a standard parabolic. Moreover $H \cdot A_Q \cdot N_Q$ is an algebraic subgroup of Q which is amenable since $A_Q N_Q$ is normal and solvable. It is also isotropically connected since if B is a maximal split torus of H, we may conjugate H so that B ends up in $M_Q \cdot A_Q$, the centralizer of A_Q. Then $B \cdot A_Q$ is a maximal **R**-split torus in H whose centralizer neets every component of H. Moreover $HA_Q N_Q$ has properly larger dimension than H unless H already contains $A_Q N_Q$, and so $H \supset A_Q \cdot N_Q$ in view of our hypotheses on H. In particular N_Q is the nilradical of H.

Now $A_Q \cdot N_Q \subset H \subset Q = M_Q \cdot A_Q \cdot N_Q$ and we conjugate H if necessary so that H has a maximal **R**-split torus B with $A_Q \subset B \subset A$ where A is our fixed maximal **R**-split torus in G. Now as H is amenable, $B \cdot N_Q$ is normal and cocompact in H, and as H is isotropically connected, $(Z(B) \cap H) \cdot N_Q = N$ where $Z(B)$ is the centralizer of B in G. Now we claim that $B = A_Q$ for if B is properly larger, then $B \cap M_Q$ is a nontrivial **R**-split torus in M_Q and we know from the above that $H = (H \cap M_Q)A_Q N_Q$ and that $(H \cap M_Q)$ is contained in the centralizer of $B \cap M_Q$. But M_Q is a reductive group with compact center and in such a group, the centralizer of a nontrivial **R**-split torus must normalize a nontrivial unipotent subgroup (in fact the unipotent radical of some parabolic subgroup). Consequently $H \cap M_Q$ normalizes a proper unipotent subgroup V of M_Q, and then $H \cdot V$ is amenable and isotropically connected and has properly larger dimension unless $H \supset V$. But this is impossible since N_Q is the unique maximal unipotent subgroup and N_Q cannot contain V.

We conclude that $B = A_Q$ and then $H = (H \cap M_Q) \cdot A_Q \cdot N_Q$ where $H \cap M_Q$ is compact. We can then conjugate H inside Q so that $H \cap M_Q \subset K_Q$, the given maximal compact subgroup of M_Q. Then $H \subset S_Q$ as desired.

Finally it is clear from the structure of S_Q that any algebraic subgroup of S_Q is isotropically connected, and hence by definition any subgroup of S_Q is isotropically connected. Now Theorem 3.2 is completely proved.

Added in proof. We thank I. Guivaréh for calling our attention to his preprint *Quelques propriétés asymptotiques des produits de matrices aleutoires* in which among many other things it is proved that H_μ (Theorem 2.7) is amenable. His proof is quite different.

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